Lecture 14 25-10-18

§ System of first order equation:

Recall that: for the second order ODE

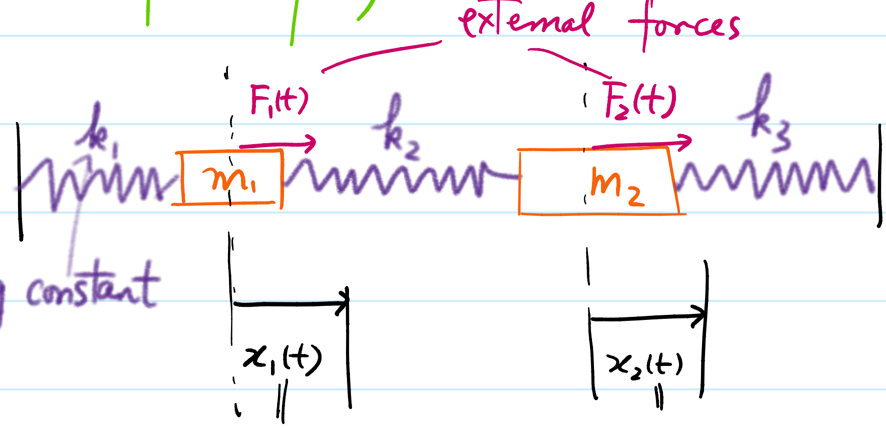
$$ay'' + by' + cy = r(t)$$

We may let $v = y'$, and it is equivalent to

$$\begin{cases} y' = v \\ av' + bv + cy = r(t) \end{cases}$$

it is a system of 1^{st} order equation with two unknown!

E.g. (real life example)



$x_1(t)$ = displacement of m_1 $\qquad$ $x_2(t)$ = displacement of m_2

$F = ma$ Equation for m_1: $\quad m_1 \dfrac{d^2x_1}{dt^2} = k_2(x_2 - x_1) - k_1 x_1 + F_1(t)$

" " for m_2: $\quad m_2 \dfrac{d^2x_2}{dt^2} = -k_2(x_2 - x_1) - k_3 x_2 + F_2(t)$

Def: A system of 1^{st} order differential equation with n dependent variables $y_1, \ldots, y_n$ defining on $I \subseteq \mathbb{R}$ is

$$\begin{aligned} y_1' &= F_1(t, y_1, \ldots, y_n) \\ y_2' &= F_2(t, y_1, \ldots, y_n) \\ &\vdots \\ y_n' &= F_n(t, y_1, \ldots, y_n) \end{aligned}$$

and a IVP is of the form:

$$y_1(t_0) = x_0, \ldots\ldots, y_n(t_0) = x_n.$$

E.g.
$$\begin{cases} y' = v \\ av' + bv + cy = r(t) \end{cases}$$
in earlier page

Let $y_1 = y, \quad y_2 = v$

then we take
$$F_1(t, y_1, y_2) = y_2$$
$$F_2(t, y_1, y_2) = \frac{-c}{a} y_1 - \frac{b}{a} y_2 + r(t)$$

Ex: Try to write the equation for the example in the previous page to a system of 1^{st} order.

Def: • A system of 1^{st} order linear differential equation of n dependent variables $y_1, \dots, y_n$ defining on I is

$$y_1' = P_{11}(t)\, y_1 + P_{12}(t)\, y_2 + \cdots + P_{1n}(t)\, y_n + r_1(t)$$
$$y_2' = P_{21}(t)\, y_1 + \cdots + P_{2n}(t)\, y_n + r_2(t)$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$y_n' = P_{n1}(t)\, y_1 + \cdots + P_{nn}(t)\, y_n + r_n(t)$$

for some $P_{11}(t) \cdots P_{nn}(t)$, $r_1(t) \dots, r_n(t)$ function defining on interval I.

<u>or if we write:</u>

$$\vec{y} = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}, \quad P(t) = \begin{pmatrix} P_{11}(t) & \cdots & P_{1n}(t) \\ \vdots & \ddots & \vdots \\ P_{n1}(t) & \cdots & P_{nn}(t) \end{pmatrix}, \quad \vec{r}(t) = \begin{pmatrix} r_1(t) \\ \vdots \\ r_n(t) \end{pmatrix}$$

then it can be rewritten as

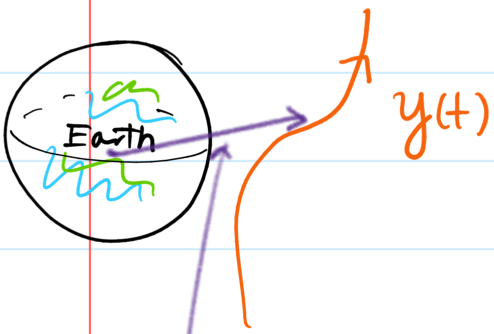
$$\vec{y}'(t) = P(t)\, \vec{y}(t) + \vec{r}(t) \quad \text{in matrix form}$$

• It is called homogeneous if $\vec{r}(t) = 0$.

Eg. $F_1(t, y_1, y_2) = y_2$, $F_2(t, y_1, y_2) = \frac{-c}{a} y_1 - \frac{b}{a} y_2 + r(t)$

$$P(t) = \begin{pmatrix} 0 & 1 \\ \frac{-c}{a} & \frac{-b}{a} \end{pmatrix}, \quad \vec{r}(t) = \begin{pmatrix} 0 \\ r(t) \end{pmatrix}$$

Eg. (Newtonian equation for gravity)



$$m\vec{y}'' = F = \frac{-GmM}{r^2}\left(\frac{\vec{y}}{\|\vec{y}\|}\right)$$

$$\|\vec{y}(t)\| = r$$

this vector at $\vec{y}(t)$ is the unit vector pointing outward

- Hence, we obtain the equation for $\vec{y} = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}$

$$\begin{pmatrix} y_1'' \\ y_2'' \\ y_3'' \end{pmatrix} = \frac{-GM}{(y_1^2+y_2^2+y_3^2)^{3/2}} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

- It is a system of 2^{nd} order ODE
- This is <u>NOT</u> a linear equation.

Def: The system of 1^{st} order linear equation is said to be has constant coefficient if $P_{ij}(t) = P_{ij}$ is a constant.

Eg. Let say we have n-th order equation:

$$y^{(n)} = F(t, y^{(0)}, \dots, y^{(n-1)})$$

we can let: $y_1 = y,\ y_2 = y',\ \dots,\ y_n = y^{(n-1)}$

and get

$$\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ \vdots \\ y_n' = F(t, y_1, \dots, y_n) \end{cases}$$

a system of 1^{st} order eq$^{\underline{n}}$.

- If $F(t, y^{(0)}, \dots, y^{(n-1)}) = -P_{n-1}(t) y^{(n-1)} - \dots - P_0(t) y + r(t)$ is a linear equation, then the matrix:

$$P(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & & \\ 0 & 0 & 1 & & & \\ 0 & & & \ddots & & \\ \vdots & & & & \ddots & \\ 0 & \cdots & & & & 1 \\ -P_0(t) & \cdots & & & & -P_{n-1}(t) \end{pmatrix}, \quad \vec{r}(t) = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ r(t) \end{pmatrix}$$

in general

Rk:
- A system of 1^{st} order ODE is the most general form! as any other system of higher order ODE can be reduced to this form!
- All the results from previous chapters can be deduced by this!

Thm: (Existence and uniqueness theorem in general)

- Let $I = (a_0, b_0)$, with $t_0 \in I$ be a point.
- Assume $F_1, \ldots, F_n$ function on $\underset{t}{(a_0,b_0)} \times \underset{y_1}{(a_1,b_1)} \times \cdots \underset{y_n}{(a_n,b_n)}$

  s.t. $F_i, \frac{\partial F_i}{\partial y_1}, \ldots, \frac{\partial F_i}{\partial y_n}$ are all continuous

  on $(a_0,b_0) \times \cdots \times (a_n,b_n)$.

- Fix $x_1 \in (a_1,b_1), \ldots, x_n \in (a_n,b_n)$.

  and consider the IVP: $y_1(t_0) = x_1, \ldots, y_n(t_0) = x_n$.

Conclusion: $\exists\, \delta > 0$, s.t. there exist an unique

solution $\vec{y}(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}$ on $(t_0 - \delta, t_0 + \delta) \subseteq I$

for the above IVP.

Rk: If we take $n = 1$, then this is the existence and uniqueness result that we have for a single 1^{st} order equation.

Thm (Existence and unique for linear equation)

For system of 1^{st} order linear equation, if $P_{11}(t) \ldots P_{1n}(t)$,

$\ldots, P_{n1}(t), \ldots, P_{nn}(t), r_1(t), \ldots, r_n(t)$ continuous on I

then $\exists!$ solution $(y_1, \ldots, y_n)^t = \vec{y}(t)$ on whole I.

for the IVP.

Prop: (superposition)

- Let $\vec{y}_1(t)$ satisfy $\vec{y}_1'(t) = P(t)\vec{y}_1(t) + \vec{r}_1(t)$
  $\vec{y}_2(t)$ satisfy $\vec{y}_2'(t) = P(t)\vec{y}_2(t) + \vec{r}_2(t)$.

  then if we let $\vec{y}(t) = c_1\vec{y}_1(t) + c_2\vec{y}_2(t)$

  satisfy: $\vec{y}'(t) = P(t)\vec{y}(t) + c_1\vec{r}_1(t) + c_2\vec{r}_2(t)$

- In particular, if $\vec{r}_1 = \vec{r}_2 = \vec{0}$, then
  $\vec{y}'(t) = P(t)\vec{y}(t)$.

Cor: If we let $\mathcal{S} = \left\{ \vec{y} = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix} \middle| \ \vec{y}'(t) = P(t)\vec{y}(t) \right\}$

then $\mathcal{S}$ is a $\mathbb{R}$-vector space ($\mathbb{C}$-vector space if $P(t)$ is $\mathbb{C}$-valued and if we look for $\mathbb{C}$-valued $\vec{y}(t)$)